## Learning progression for multi-digit multiplication

1. Does the student know that the concept of multiplication is repeated adding or skip counting – finding the total number of objects in a set of equal size groups. Is the students able to represent situations involving groups of equal size with objects, words and symbols. (N.MR.02.16)

   Can the student solve a simple problem such as *Megan has 7 bags of cookies. There are 5 cookies in each bag. How many cookies does Megan have all together?* A student could skip-count by 5's to get this answer.

2. Does the student know and draw on basic facts and other number relationships? (Find products fluently up to 10 x 10, N.FL.03.11)

   Basic number combination facts are learned easily over time by most children, when their teachers use a program that builds on students' innate problem-solving abilities and allows them to develop multiple strategies for number combinations. The use of flash cards alone does not result in fluency with number facts for many students. They need to build familiarity with fact families through exposure to the facts in many problem situations and through the development of increasingly abstracted strategies for processing more difficult number combinations. **See the "fluency" packet for more details.** Also see "No Need to Memorize" all the facts – a summary of strategies: http://naturalmath.com/mult/mult13.html

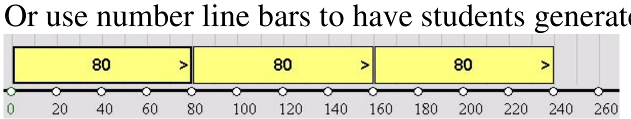
3. Does the student know how to multiply by 10's and 100's? (N.FL.03.13)

   To help students learn this, give them a calculator to work out several problems that involve multiplying by 10's and 100's, or multiples of 10's and 100's. Let them look for the pattern. Before they know the "trick," they can skip count on an appropriate number line to get the answer.

   10 x 3     40 x 6     300 x 4     400 x 7     20 x 40

   For example: 80 x 3 is the same as 8 x 3 with a zero added. One zero - coming from the 80.

   Or use number line bars to have students generate the pattern: If 8 x 3 is 24, what's 80 x 3?

4. Can the student use number sense to estimate the result of multiplying?

   Rounding: 26 x 4 is close to 25 x 4, which is like using money (4 quarters). Compensating involves adjusting each factor, for example, 95 x 11 could be estimated using 100 x 10. Good for checking the reasonableness of answers.

5. If the problem is stated in a real-world context, does the context give the student a clue about how to solve the problem directly?

   Contexts such as price, rates, comparisons, or simply "6 groups of 54" sometimes help the student visualize the problem – for example, "If you travel 54 miles each day for 6 days, how far would you travel altogether?" The student might round down to 50 miles and easily figure 50 x 6 = 300. Let the student make a drawing, if that helps. **See the "fluency" packet for a list of various problem types and contexts.**

6. Does the student understand graphical representations like area or array representations? (N.MR.02.14)

   Array models are like desks lined up in a classroom: 5 rows of 7 desks is 35 desks. Area models are like brownies in a pan: 4 rows of 6 brownies makes 24 brownies. The colorful representation on the last page shows how this works with 10's and 100's.

   To help students learn to use arrays and area models, use the NCTM Illuminations lessons on multiplication, Lesson 3, at http://illuminations.nctm.org/LessonDetail.aspx?ID=U109.
   - Lesson 1: skip counting on the number line (more number line games: 1, 2)
   - Lesson 2: learning the concept of groups
   - Lesson 3: rows and columns – the array concept of multiplication
   - Lesson 4: pan balance and product game

   A simulations of the area model for single digit and multi-digit multiplication can be found here: http://enlvm.usu.edu/ma/nav/activity.jsp?sid=nlvm&cid=2_1&lid=192

   An area model for division with remainders can be found here: http://nlvm.usu.edu/en/nav/frames_asid_193_g_2_t_1.html?from=category_g_2_t_1.html

7. Does the student understand the distributive property with simple problems? (N.ME.04.09 Multiply two-digit numbers by 2, 3, 4, and 5, using the distributive property…)

   e.g. 54 • 5 = (50 • 5) + (4 • 5) = 250 + 20 = 270 This is the concept behind both the partial product algorithm (which makes much more sense to students) and the standard algorithm. In both cases, students multiply the ones, ten, hundreds, etc. separately. This problem is easily worked out mentally when a student understands how to use the distributive property.

8. Does the student have any strategies for solving the problem, such as creating more easily managed sub-problems?

   This might involve an intuitive use of factors, for example: 54 x 6 = 54 x 2 x 3 = 108 + 108 + 108 = 324. The use of this strategy depends on the problem. In this case, doubling 54 was relatively easy, and adding it 3 times was also relatively easy. Give the student simple problems like this to try. For 54 x 23, the student might simplify the problem like this:
   (54 • 10) + (54 • 10) + 108 + 54 This is a good build-up to the partial product method.

9. Can the student use alternative algorithms like the partial product method? (N.FL.04.10):

   The partial product method is shown at the right. The lattice method, and a variation of the partial product method, are shown at the end.

```
  54
x  6
----
 300
  24
----
 324
```

```
   54
x  23
-----
   12
  150
   80
 1000
-----
 1242
```

10. Can the student multiply fluently any whole number by a one-digit number, and a three-digit number by a two-digit number? (N.FL.04.10)

    Any whole number by a single digit; a three digit number by a two digit number. Either the tradition algorithm or the lattice method are acceptable procedures.

11. Can the student avoid typical errors when using the standard algorithm? (N.FL.05.04)

    Some students know most of the steps of the standard algorithm but make typical errors, which can be corrected with direct instruction on the concepts behind the algorithm. Typical errors using the standard algorithm involve not accounting correctly for place value. These are some examples of common errors:

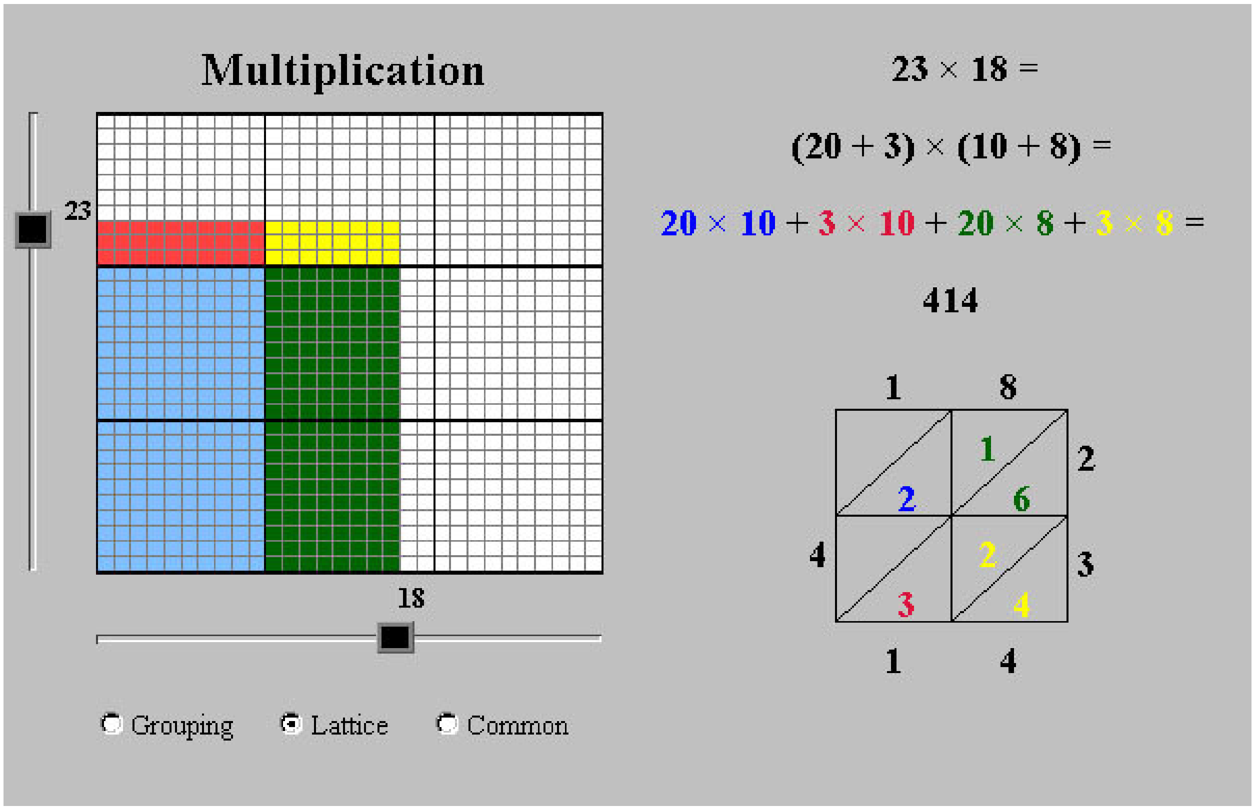
```
                  25        25
  54             x43       x73
 x 6             615        75
3024            820        175
                8815       250
```

```
 2
 57
x 4
----
288
```

Another error involves not knowing what to do with the number that is "carried." In this example, the student multiplied 7 x 4 and got 28. He put down the 8 and wrote the 2 above the ten's place. But then he added the 2 and the 5 *before* multiplying by 4. Using the partial product method (see above and below) can help correct this.

There is no need to teach the standard algorithm if the student makes too many errors with it – students can be fluent with the partial product method. Or use the lattice method if you insist on something more compact. It is less error-prone than the traditional algorithm. (See graphic below for lattice method. It is the same as the standard algorithm, only on the diagonal.)

**Area multiplication showing the lattice method**
http://nlvm.usu.edu/en/nav/frames_asid_192_g_2_t_1.html
Color coding is used to point out the distributive property.




**Abbreviated array model:**

| | 10 | +8 |
|---|---|---|
| 3 | 30 | 24 |
| + | | |
| 20 | 200 | 160 |

**Partial product method with variation:**

```
   18
x  23
   24
   30
  160
  200
  414
```

| 18<br>x23 | 10 | 8 | |
|---|---|---|---|
| 20 | 200 | 160 | 360 |
| 3 | 30 | 24 | +54 |
| | | | 414 |

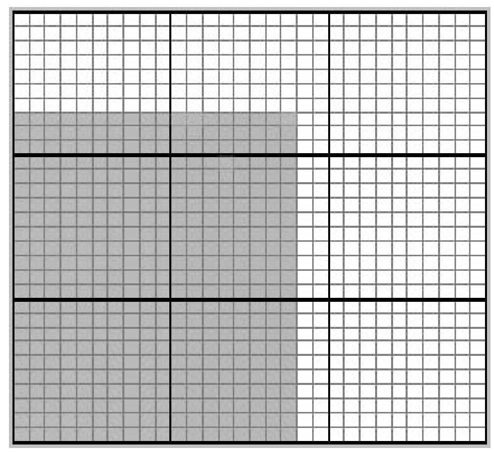
**Multi-digit Multiplication Diagnostic Problems**

1. How much is 25 x 10? _______________

2. 43 x 25 = _____________ **Show all your work.** You can solve this any way you want.

3. How much is 78 x 1000? ______________

4. What multiplication does this picture represent?



   What's the answer to this multiplication?

   How did you get your answer?

5. If one lamp costs $45, how much do 12 lamps cost? **Show how you figured this out and circle your answer.**

6. Mike has 27 boxes of trading cards. Each box has 68 cards in it. How many cards does Mike have altogether? **Show all your work and circle your answer.**

7. Solve these two problems about a bus trip to Chicago:

   57 people were taking a trip to Chicago on buses. 3 buses were hired to take them there. The distance to Chicago is 248 miles. Half-way to Chicago, one of the buses broke down. Everyone could fit on the remaining two buses except for 5 people. What is the maximum number of people that one of the buses could hold?

   The 5 people who couldn't fit on the two buses hired a small plane to take them the rest of the way to Chicago. The plane cost $13 per mile. How much did the plane ride cost?

| Concrete objects (Children's mathematics) | Pictures (Graphic representations) | Symbols (Symbolic representations) |
|---|---|---|

## Concrete objects (Children's mathematics)

*Bugs have six legs each. How many legs are there on three bugs?*

*You have six cookies and want to share them equally with two friends. How many cookies does each friend get?*

Children use many different strategies to solve problems like these. Children may count or skip count to get an answer, or use manipulatives to aid in counting, or draw a picture. At this stage, let them use whatever strategy they can, and let them learn from each other.

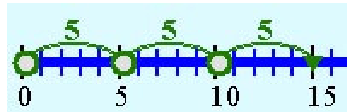
**Place value:** Use bundles of 10's and some ones in simple multiplication problems to learn place value:

*Each box of crayons has 10 crayons in it. Karen has 3 boxes of crayons and 4 extra crayons. How many is this? Write this number.*

**Distributive property:** Use base ten blocks to model 16 x 12, showing the distributive property (16 x 10 + 16 x 2; 10 x 10 + 6 x 10 + 10 x 2 + 6 x 2) by regrouping the blocks.

## Pictures (Graphic representations)

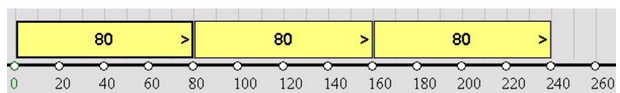
Skip counting on the number line:



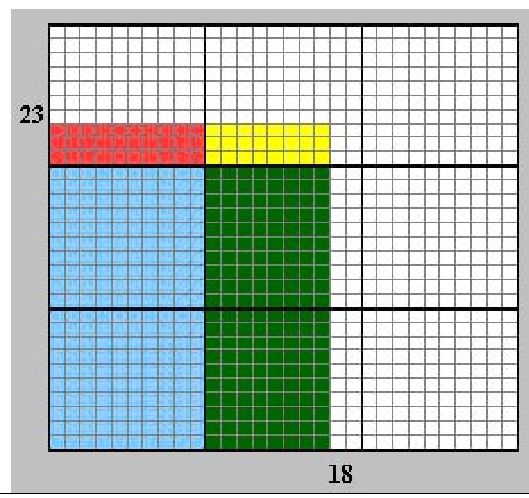

**Multi-digit multiplication:**

Multiplying by 10's and 100's: Use number line bars, have students generate the pattern:

If 8 x 3 is 24, what's 80 x 3? What simple procedure can we use to do these quickly?




**Distributive property** using area model:




## Symbols (Symbolic representations)

**Distributive property:**

23 x 18 = 20(10 + 8) + 3(10 + 8)

**The partial product method:**

| 18 x 23 | 10 | 8 | |
|---|---|---|---|
| 20 | 200 | 160 | 360 |
| 3 | 30 | 24 | 54 |
| | | | 414 |

| | | |
|---|---|---|
| | 2 | 3 |
| x | 1 | 8 |
| | 2 | 4 |
| 1 | 6 | 0 |
| | 3 | 0 |
| 2 | 0 | 0 |
| 4 | 1 | 4 |

**Other methods:**

Lattice method (also involves regrouping but no "little numbers" in the multiplication stage)